Fast MOR-based Approach to Uncertainty Quantification in Transcranial Magnetic Stimulation

Lorenzo Codecasa[1], *Member, IEEE*, Luca Di Rienzo[1], *Senior Member, IEEE*, Konstantin Weise[2], *Student Member, IEEE*, Stefanie Gross[3], and Jens Haueisen[3], *Member, IEEE*

[1]Politecnico di Milano, Dipartimento di Elettronica, Informazione e Bioingegneria, Milano, Italy, lorenzo.codecasa@polimi.it, luca.dirienzo@polimi.it
[2]TU Ilmenau, Department of Advanced Electromagnetics, Ilmenau, Germany, konstantin.weise@tu-ilmenau.de
[3]TU Ilmenau, Institute of Biomedical Engineering and Informatics, Ilmenau, Germany, stefanie.gross@tu-ilmenau.de, jens.haueisen@tu-ilmenau.de

We propose a Model Order Reduction approach to the uncertainty quantification in Transcranial Magnetic Stimulation and compare it with a standard non-intrusive PCE approach. Thanks to the new algorithm the computational time is reduced by more than two orders of magnitude with respect to standard non-intrusive approaches, at comparable accuracy.

Index Terms—**Magnetic stimulation, Model Order Reduction, Polynomial Chaos Expansion, Uncertainty.**

I. Introduction

THE quantification of uncertainty can play an important role in Transcranial Magnetic Stimulation (TMS) [1] and there is an essential need for more effective techniques due to the increasing model complexity. In most cases, Monte Carlo methods are too expensive and techniques based on Polynomial Chaos Expanion (PCE) are favourable.

II. TMS Deterministic Modeling

In the deterministic modeling phase we use a realistic head model [2], which contains five different tissues, namely scalp, skin, cerebrospinal fluid (CSF), grey matter (GM) and white matter (WM). The excitation coil is a Magstim 70 mm double coil with 9 windings and is placed above the motor cortex area M1 (Brodman area 4) at a distance of 4 mm from the skalp. The coil is approximated by means of 2712 magnetic dipoles constituted in three layers [3]. The electromagnetic problem at hand is simplified due to the low electrical conductivities and moderate excitation frequencies which are in the range of $2-3$ kHz so that the secondary magnetic field from the induced eddy currents can be neglected. In this way, the magnetic field can be expressed in terms of the magnetic vector potential $\mathbf{a}_c$ produced by the excitation coil ($\mathbf{b}_c = \nabla \times \mathbf{a}_c$, $\nabla \cdot \mathbf{a}_c = 0$). Considering the current conservation law, this reduces to solve the following equation at angular frequency ω with Neumann conditions on the boundary $\partial\Omega$ of the spatial domain Ω

$$\nabla \cdot (-\sigma(\mathbf{r},\mathbf{p})\nabla\varphi(\mathbf{r},\mathbf{p})) = i\omega\, \nabla \cdot (\sigma(\mathbf{r},\mathbf{p})\mathbf{a}_c(\mathbf{r})), \quad (1)$$

in which the unknown $\varphi(\mathbf{r},\mathbf{p})$ is the electric potential, $\mathbf{a}_c(\mathbf{r})$ is the known magnetic vector potential, and $\sigma(\mathbf{r},\mathbf{p})$ is the electric conductivity; the latter can be assumed to be a *linear* combination of the n parameters p_i, forming vector $\mathbf{p}$

$$\sigma(\mathbf{r},\mathbf{p}) = \sigma_0(\mathbf{r}) + \sum_{i=1}^{n} \sigma_i(\mathbf{r})\, p_i. \quad (2)$$

III. TMS Stochastic Modeling

The electrical conductivities of scalp and skin are modelled as deterministic since they poorly affect the induced electric field inside the human brain. On the other hand, the conductivities of CSF, GM, and WM show a wide spread across individuals and measurements [4] and are then modelled as uniform distributed random variables with the limits given in the caption of Fig. 1. In a stochastic analysis, the parameters forming vector $\mathbf{p}$ are assumed to be random variables. Then, applying PCE, $\varphi(\mathbf{r},\mathbf{p})$ is approximated in the form

$$\varphi(\mathbf{r},\mathbf{p}) = \sum_{|\boldsymbol{\alpha}|\leq M} \phi_{\boldsymbol{\alpha}}(\mathbf{r})\psi_{\boldsymbol{\alpha}}(\mathbf{p}), \quad (3)$$

in which $\boldsymbol{\alpha}$ are multi-indices of n elements and $\psi_{\boldsymbol{\alpha}}(\mathbf{p})$ are polynomials of degrees less than a chosen M, forming an orthonormal basis in the probability space of random variable p_i. In standard non-intrusive PCE approaches, commonly adopted as the most efficient alternative to Monte Carlo technique, functions $\phi_{\boldsymbol{\alpha}}(\mathbf{r})$ are reconstructed from the solutions $\varphi(\mathbf{r},\mathbf{p})$ of the deterministic problems $\mathcal{D}$ for all values of $\mathbf{p}$ in a proper set $\mathcal{G}$. However, even using sparse-grids [5], the set $\mathcal{G}$ becomes very large when the number n of parameters or the polynomial degree M increases. Thus the number of deterministic problems to be solved also becomes very large.

A. The Model Order Reduction Approach

Hereinafter the alternative Algorithm 1 is proposed which constructs a reduced order model tailored to PCE analysis by solving a much smaller number of deterministic problems with respect to the non-intrusive approaches. The PCE expansion of the solution to the original problem is then obtained from such reduced order model. In the algorithm, at step 1, the deterministic problem $\mathcal{D}$ is numerically approximated, for the selected values of $\mathbf{p}$, by a discretization method. At step 2 an

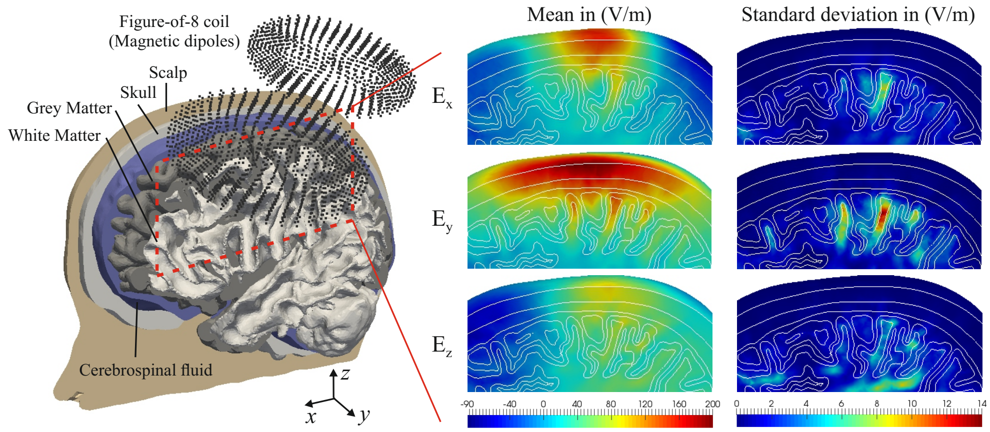

Fig. 1. Mean value and standard deviation of the induced electric field components in the sagittal plane under the excitation coil determined with the MOR approach. The following values for the conductivity are used (S/m): for scalp 0.34, for skull 0.025, for CSF between 1.432 and 2.148, for grey matter between 0.153 and 0.573, and for white matter between 0.094 and 0.334.

Algorithm 1: MOR-based algorithm

Set $k := 0$ (dimension of the reduced model)
Set $\vartheta := 0$ (norm of the residual)
Set linear space $\mathcal{S}_0 := \emptyset$
Choose vector $\mathbf{p}$ in $\mathcal{G}$
repeat
    Set $k := k + 1$
1   Solve problem $\mathcal{D}$ for $\varphi(\mathbf{r}, \mathbf{p})$
2   Generate an orthonormal basis of the linear space $\mathcal{S}_k$ spanned by $\mathcal{S}_{k-1}$ and $\varphi(\mathbf{r}, \mathbf{p})$
3   Generate reduced order model $\mathcal{R}_k(\mathbf{p})$, projecting problem $\mathcal{D}$ onto space $\mathcal{S}_k$
    for all $\mathbf{q} \in \mathcal{G}$ **do**
4   Solve reduced order model $\mathcal{R}_k(\mathbf{q})$ and approximate $\varphi(\mathbf{r}, \mathbf{q})$ by $\hat{\varphi}(\mathbf{r}, \mathbf{q})$
5   Estimate the approximation error η
        if $\eta > \vartheta$ **then**
            Set $\vartheta := \eta$
6   Set $\mathbf{p} := \mathbf{q}$
until $\vartheta > \varepsilon$
7 Determine the PCE expansion of the solution to the reduced order model $\mathcal{R}_k(\mathbf{p})$ and reconstruct the PCE expansion of $\varphi(\mathbf{r}, \mathbf{p})$

orthonormal basis of space $\mathcal{S}_k$ is generated, computing a set of functions $v_h(\mathbf{r})$, with $h = 1, \ldots, k$, spanning all functions $\varphi(\mathbf{r}, \mathbf{p})$ computed at step 1. At step 3 the reduced order model $\mathcal{R}_k(\mathbf{p})$, obtained by projecting (1) and boundary conditions over the space spanned by functions $v_h(\mathbf{r})$, with $h = 1, \ldots, k$, has the form

$$\left(\hat{\mathbf{S}}_0 + \sum_{i=1}^{n} p_i \hat{\mathbf{S}}_i\right) \hat{\boldsymbol{x}}(\mathbf{p}) = \left(\hat{\mathbf{R}}_0 + \sum_{i=1}^{n} p_i \hat{\mathbf{R}}_i\right)$$

in which $\hat{\mathbf{S}}_i$ are square matrices of dimension k

$$\hat{\mathbf{S}}_i = \left[\int_\Omega \nabla v_h(\mathbf{r}) \cdot \sigma_i(\mathbf{r}) \nabla v_l(\mathbf{r})\, d\mathbf{r}\right]$$

and $\hat{\mathbf{R}}_i$ are column vectors of k rows

$$\hat{\mathbf{R}}_i = \left[-i\omega \int_\Omega \nabla v_h(\mathbf{r}) \cdot \sigma_i(\mathbf{r}) \mathbf{a}(\mathbf{r})\, d\mathbf{r}\right],$$

with $i = 0, \ldots, n$. The elements $\hat{x}_h$ of vector $\hat{\boldsymbol{x}}$ allow to approximate the solution $\varphi(\mathbf{r}, \mathbf{p})$ to problem $\mathcal{D}$ as (step 4)

$$\hat{\varphi}(\mathbf{r}, \mathbf{p}) = \sum_{h=1}^{k} v_h(\mathbf{r}) \hat{x}_h(\mathbf{p}). \tag{4}$$

At step 5, η represents the residual when $\varphi(\mathbf{r}, \mathbf{q})$ is substituted by $\hat{\varphi}(\mathbf{r}, \mathbf{q})$ in $\mathcal{D}$. At step 6, the value of $\mathbf{q}$ in $\mathcal{G}$ maximizing the value of η becomes the candidate $\mathbf{p}$ for solving the deterministic problem $\mathcal{D}$ at next step 1. At step 7, a PCE approach is applied to the reduced order model $\mathcal{R}_k$. From the PCE expansion of $\hat{\boldsymbol{x}}(\mathbf{p})$, derived in negligible time, the PCE expansion of $\hat{\varphi}(\mathbf{r}, \mathbf{p})$ approximating the PCE of $\varphi(\mathbf{r}, \mathbf{p})$ is obtained from (4).

B. Numerical Results

Each deterministic problem is discretized using approximately $5 \cdot 10^5$ linear finite elements and the full grid $\mathcal{G}$ is composed by 125 nodes. The MOR-based and the non-intrusive methods require respectively 80 s using a 2.3 GHz Intel Core 7 PC and 480 min using a 3.1 Ghz Intel Core i5-3450. The relative difference of both the mean and the standard deviation of each component of electric field, provided by the two approaches, is about 0.1% in the maximum norm. The used PCE order is $M = 5$.

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